Fuzzy Inf. Eng. (2016) 8: 1–15

HOSTED BY

ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Information and Engineering

http://www.elsevier.com/locate/fiae

Fuzzy Information and Engineering

An International Journal

ORIGINAL ARTICLE

Soft Ternary Semirings

CrossMark

S. Kar · A. Shikari

Received: 21 October 2014/ Revised: 30 October 2015/
Accepted: 12 January 2016/

Abstract In this paper, we introduce the notion of soft ternary semiring by using the concept of soft set theory. Besides, we characterize the notions of regularity and intra-regularity in soft ternary semiring by using different soft (left, lateral, right, quasi, bi) ideals of soft ternary semirings.

Keywords Ternary semiring · Soft ternary semiring · Soft ideal · Regular soft ternary semiring · Intra-regular soft ternary semiring

1. Introduction

Science and technology are now featured with complex processes and phenomena for which complete information is not always available to us. For such cases, mathematical models are developed to handle various types of systems containing uncertainty. In our daily life, uncertainty appears in common phenomena because our surroundings is full of uncertainties. Therefore, it is quiet natural for us to model this uncertainty prevailing in physical world. A large number of these models is based on an extension of ordinary set theories such as fuzzy sets, intuitionistic fuzzy sets, soft sets etc. Among these there are mainly three theories namely, theory of probability, theory of fuzzy sets and interval mathematics, which are used to model such situations. But all these theories have certain limitations to describe uncertain problems in a lucid manner. Theory of probability can deal only stochastically stable phenomena

S. Kar (✉) · A. Shikari
Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India
email: karsukhendu@yahoo.co.in
Peer review under responsibility of Fuzzy Information and Engineering Branch of the Operations Research Society of China.

http://dx.doi.org/10.1016/j.fiae.2016.03.001

which need a large number of trials. Interval mathematics are also not adequate for problems with different level of uncertainties. Most of the problems in engineering science, medical science, social science, environmental science etc. have various uncertainties. The problems in system identification involve characteristics which are essentially non probabilistic in nature. Perhaps the most appropriate theory to tackle problems of uncertainty is fuzzy set theory introduced by Zadeh [18]. Intuitionistic fuzzy sets have also drawn the attention of many researchers in the last decades. This is mainly due to the fact that intuitionistic fuzzy sets are consistent with human behavior by reflecting and modeling the hesitancy present in real-life. In fact, the fuzzy sets give the degree of membership of an element in a given set, while the intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. But major difficulty arising in these theories are probably due to the inadequacy of parameters. The theory of soft set seems to be more adequate and it is a general mathematical tool for dealing with uncertain, fuzzy and not clearly defined objects. The notion of soft set was developed by D. Molodtsov [14] by involving enough parameters so that it will be helpful for modeling uncertainty. Soft set theory [12, 13] has a rich potential for applications in our day to day problems. It can be considered as a technique for solving real life problems like medical diagnosis, decision-making, data analysis and decision support systems. Soft set theory can be applied to many areas of engineering science, social science, medical science etc. with great efficiency.

On the other hand, there is a large literature dealing with ternary algebra. In 1971, W.G. Lister [11] introduced the notion of ternary ring and provided some types of representations of ternary ring. The notion of ternary semiring was first introduced by Dutta and Kar [5] as a generalization of ternary ring introduced by Lister [11]. A great deal of research has been done and is being done in the area of ternary semiring [7, 9, 10]. Besides the algebraic soft structures were studied by many researchers [1-4, 15-17] with the help of soft set theory, the notion of soft semirings were introduced by Feng et al. [8] in 2008. Our approach to study soft ternary semiring is motivated by the above soft algebraic structures.

The main purpose of this paper is to introduce the algebraic structure of soft ternary semiring which extends the notion of ternary semiring in soft set setting. In this paper, we define the notion of soft ternary semiring by using the concept of soft set theory and characterize the notion of regularity and intra-regularity in soft ternary semiring by using different soft (left, lateral, right, quasi, bi) ideals of soft ternary semirings.

2. Soft Sets and Ternary Semirings

In this section, we recall some basic notions of soft sets and ternary semirings which will be needed for characterizing soft ternary semirings.

Definition 2.1 [8] *Let U be an initial universal set and E be a set of parameters. Suppose that $\mathcal{P}(U)$ denotes the power set of U and A be a non-empty subset of E. A pair (η, A) is called a soft set over U, where $\eta : A \longrightarrow \mathcal{P}(U)$ is a mapping.*

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\epsilon \in A$, $\eta(\epsilon)$ may be considered as the set of ϵ-approximate elements of the soft set (η, A).

Maji et al. [13] introduced and investigated several binary operations in soft sets such as intersection, union, And-operation and OR-operation which are as follows:

Definition 2.2 [13] *Let (η, A) and (γ, B) be two soft sets over a common universe U. Then the intersection of (η, A) and (γ, B) is defined to be the soft set (ϑ, C) satisfying the following conditions:*

(i) $C = A \cap B$,

(ii) *for all $e \in C$, $\vartheta(e) = \eta(e)$* or *$\gamma(e)$ (as both are the same set).*

In this case, we write $(\eta, A) \,\widetilde{\cap}\, (\gamma, B) = (\vartheta, C)$.

Let (η, A) and (γ, B) be two soft sets over a common universe U. The union of (η, A) and (γ, B) is defined to be the soft set (ϑ, C) satisfying the following conditions:

(i) $C = A \cup B$,

(ii) *for all* $e \in C$, $\vartheta(e) = \begin{cases} \eta(e), & \text{if } e \in A \setminus B, \\ \gamma(e), & \text{if } e \in B \setminus A, \\ \eta(e) \cup \gamma(e), & \text{if } e \in A \cap B. \end{cases}$

In this case, we will write $(\eta, A) \,\widetilde{\cup}\, (\gamma, B) = (\vartheta, C)$.

Definition 2.3 [13] *If (η, A) and (γ, B) are two soft sets over U, then "(η, A) AND (γ, B)" denoted by $(\eta, A) \,\widetilde{\wedge}\, (\gamma, B)$ is defined by $(\eta, A) \,\widetilde{\wedge}\, (\gamma, B) = (\vartheta, A \times B)$, where $\vartheta(x, y) = \eta(x) \cap \gamma(y)$ for all $(x, y) \in A \times B$.*

If (η, A) and (γ, B) are two soft sets over U, then "(η, A) OR (γ, B)" denoted by $(\eta, A) \,\widetilde{\vee}\, (\gamma, B)$ is defined by $(\eta, A) \,\widetilde{\vee}\, (\gamma, B) = (\vartheta, A \times B)$, where $\vartheta(x, y) = \eta(x) \cup \gamma(y)$ for all $(x, y) \in A \times B$.

Definition 2.4 [13] *For two soft sets (η, A) and (γ, B) over U, we say that (η, A) is a soft subset of (γ, B) denoted by $(\eta, A) \,\widetilde{\subset}\, (\gamma, B)$ if it satisfies:*

(i) $A \subset B$,

(ii) *for every $\varepsilon \in A$, $\eta(\varepsilon)$ and $\gamma(\varepsilon)$ are identical approximations.*

In contrast with Definition 2.2 of soft set intersections, alternatively Feng et al. [8] defined and used the following binary operation, called the bi-intersection of two soft sets.

Definition 2.5 [8] *Let (α, A) and (β, B) be two soft sets over U. Then the bi-intersection of them is defined as (γ, C), where $C = A \cap B$ and $\gamma : C \to \mathcal{P}(U)$ is a mapping given by $\gamma(x) = \alpha(x) \cap \beta(x)$ for all $x \in C$. This is denoted by $(\alpha, A) \,\widetilde{\sqcap}\, (\beta, B) = (\gamma, C)$.*

Definition 2.6 [5] *A non-empty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:*

(i) $(abc)de = a(bcd)e = ab(cde)$,

(ii) $(a + b)cd = acd + bcd$,

(iii) $a(b + c)d = abd + acd$,

(iv) $ab(c + d) = abc + abd$ *for all* $a, b, c, d, e \in S$.

Example 2.1 Let $\mathbb{Z}^-$ be the set of all negative integers. Then with the usual binary addition and ternary multiplication, $\mathbb{Z}^-$ forms a ternary semiring.

Definition 2.7 [5] *A ternary semiring* S *is called commutative if* $x_1x_2x_3 = x_{\sigma(1)}x_{\sigma(2)} x_{\sigma(3)}$ *for every permutation* σ *of* $\{1, 2, 3\}$ *and* $x_1, x_2, x_3 \in S$.

Definition 2.8 [5] *An additive subsemigroup* I *of a ternary semiring* S *is called*

(i) *left ideal of* S *if* $SSI \subseteq I$,

(ii) *lateral ideal of* S *if* $SIS \subseteq I$,

(iii) *right ideal of* S *if* $ISS \subseteq I$,

(iv) *ideal of* S *if* I *is a left, a lateral and a right ideal of* S.

An ideal I *of a ternary semiring* S *is called a proper ideal of* S *if* $I \neq S$.

Definition 2.9 [5] *Ternary semiring* S *is said to be regular if for each* $a \in S$, *there exists an element* $x \in S$ *such that* $a = axa$.

Theorem 2.1 [6] *A ternary semiring* S *is a regular ternary semiring if and only if* $R \cap M \cap L = RML$ *for every right ideal* R, *lateral ideal* M *and left ideal* L *of* S.

Definition 2.10 [7] *A ternary semiring* S *is said to be intra-regular if for all* $x \in S$, *there exist* $a_i, b_i \in S$ *such that* $x = \sum_{i=1}^{n} a_i x^3 b_i$ *for* $n \in \mathbb{N}$.

Theorem 2.2 [7] *A ternary semiring* S *is intra-regular if and only if* $L \cap M \cap R \subseteq LMR$ *for every left ideal* L, *lateral ideal* M *and right ideal* R *of* S.

3. Soft Ternary Semirings

3.1. Soft Ideals of Soft Ternary Semirings

Suppose S is a ternary semiring and A is any nonempty set. Let ρ be an arbitrary binary relation between an element of A and an element of S, that is, ρ is a subset of $A \times S$ without otherwise specified. A set-valued function $\eta : A \longrightarrow \mathcal{P}(S)$ can be defined as $\eta(x) = \{y \in S \mid (x, y) \in \rho\}$ for all $x \in A$. The pair (η, A) is then a soft set over S, which has been derived from the relation ρ.

The concept of support has been derived for fuzzy sets in the literature. Here we define a similar concept for soft sets. For a soft set (η, A), the set $\mathrm{Supp}(\eta, A) = \{x \in A \mid \eta(x) \neq \phi\}$ is called the support of the soft set (η, A). Thus a null set is indeed a soft set with an empty support, and we say that a soft set (η, A) is non-null if Supp $(\eta, A) \neq \phi$.

Definition 3.1 *Let (η, A) be a non null soft set over a ternary semiring S. Then (η, A) is called a soft ternary semiring over S if $\eta(x)$ is a ternary subsemiring of S for all* $x \in \mathrm{Supp}(\eta, A)$.

Example 3.1 Consider the ternary semiring $\mathbb{Z}_0^-$ w.r.t. the usual binary addition and the ternary multiplication. Now we consider the soft set $(\eta, \mathbb{N}_0)$ over the ternary semiring $\mathbb{Z}_0^-$ by the function $\eta : \mathbb{N}_0 \to \mathcal{P}(\mathbb{Z}_0^-)$, where $\eta(n) = n\mathbb{Z}_0^-$ and $n \in \mathbb{N}_0$. As $n\mathbb{Z}_0^-$ is a ternary subsemiring of $\mathbb{Z}_0^-$ for all $n \in \mathbb{N}_0$, so $(\eta, \mathbb{N}_0)$ is a soft ternary semiring over $\mathbb{Z}_0^-$.

Example 3.2 Consider the ternary semiring $M_2(\mathbb{Z}_0^-)$ with the usual binary addition and the ternary multiplication of matrices. Define $\eta : \mathbb{N}_0 \to \mathcal{P}(M_2(\mathbb{Z}_0^-))$ by

$$\eta(n) = \begin{pmatrix} nk_1 & nk_2 \\ nk_3 & nk_4 \end{pmatrix},$$

where $k_1, k_2, k_3, k_4 \in \mathbb{Z}_0^-, n \in \mathbb{N}_0$. Here $\eta(n)$ is a ternary subsemiring of the ternary semiring $M_2(\mathbb{Z}_0^-)$ for all $n \in \mathrm{Supp}(\eta, \mathbb{N}_0)$. So $(\eta, \mathbb{N}_0)$ is soft ternary semiring over the ternary semiring $M_2(\mathbb{Z}_0^-)$.

Definition 3.2 *Let (η_1, A_1) and (η_2, A_2) be two soft subsets of a soft ternary semiring (η, A) over a ternary semiring S. Now addition of the two soft subsets is defined by*

$$(\eta_1, A_1) \oplus (\eta_2, A_2) = (\alpha, B),$$

where $B = A_1 \cap A_2$ and $\alpha(x) = \eta_1(x) + \eta_2(x)$ for all $x \in \mathrm{Supp}(\alpha, B)$.

Definition 3.3 *Let (η_1, A_1), (η_2, A_2) and (η_3, A_3) be three soft subsets of a soft ternary semiring (η, A) over a ternary semiring S. Now the ternary multiplication of the soft subsets is defined by*

$$(\eta_1, A_1)\widehat{\odot}(\eta_2, A_2)\widehat{\odot}(\eta_3, A_3) = (\alpha, B),$$

where $B = A_1 \cap A_2 \cap A_3$ and $\alpha(x) = \eta_1(x)\eta_2(x)\eta_3(x)$ for all $x \in \mathrm{Supp}(\alpha, B)$.

Theorem 3.1 *Let (α, A) and (β, B) be two soft ternary semirings over S. Then the soft set $(\alpha, A) \widetilde{\wedge} (\beta, B)$ is a soft ternary semiring over S if it is non-null.*

Definition 3.4 *Let (α, A) and (β, B) be two soft ternary semirings over a ternary semiring S. Then the soft ternary semiring (β, B) is called a soft ternary subsemiring of (α, A) if*

(i) $B \subset A$,

(ii) *$\beta(x)$ is a ternary subsemiring of $\alpha(x)$ for all $x \in \mathrm{Supp}(\beta, B)$.*

From Definition 3.4, one can easily show that if (β, B) is a soft ternary subsemiring of (α, A), then $\mathrm{Supp}(\beta, B) \subset \mathrm{Supp}(\alpha, A)$.

Example 3.3 Consider the soft ternary semiring $(\eta, \mathbb{N}_0)$ as in Example 3.1. Now we consider $(\gamma, 2\mathbb{N}_0)$, where $\gamma : 2\mathbb{N}_0 \to \mathcal{P}(\mathbb{Z}_0^-)$ is defined by $\gamma(m) = m\mathbb{Z}_0^-$ for all

$m \in 2\mathbb{N}_0$. Then $2\mathbb{N}_0 \subseteq \mathbb{N}_0$ and $\gamma(x) = \eta(x)$ for all $x \in 2\mathbb{N}_0$. So $(\gamma, 2\mathbb{N}_0)$ is a soft ternary subsemiring of the soft ternary semiring $(\eta, \mathbb{N}_0)$.

Proposition 3.1 *Let (α, A) and (β, B) be two soft ternary semirings over a ternary semiring S. Then we have the following:*

(i) *The bi-intersection $(\alpha, A) \,\widetilde{\sqcap}\, (\beta, B)$ is a soft ternary semiring over S if it is non-null.*

(ii) *If $\beta(x) \subset \alpha(x)$ for all $x \in A$, then (β, B) is a soft ternary subsemiring of (α, A).*

(iii) *$(\alpha, A) \,\widetilde{\sqcap}\, (\beta, B)$ is a soft ternary subsemiring of both (α, A) and (β, B) if it is non-null.*

Definition 3.5 *Let $(\alpha_i, A_i)_{i\in I}$ be a nonempty family of soft sets over a common universe U. The bi-intersection of those soft sets is defined to be the soft set (β, B) such that $B = \bigcap_{i\in I} A_i$, and $\beta(x) = \bigcap_{i\in I} \alpha_i(x)$ for all $x \in B$. In this case, we denote $\widetilde{\prod}_{i\in I}(\alpha_i, A_i) = (\beta, B)$.*

Definition 3.6 *Let $(\alpha_i, A_i)_{i\in I}$ be a nonempty family of soft sets over a common universe U. The AND-soft set $\widetilde{\bigwedge_{i\in I}}(\alpha_i, A_i)$ of the soft sets $(\alpha_i, A_i)_{i\in I}$ is defined to be the soft set (β, B) such that $B = \prod_{i\in I} A_i$, and $\beta(x) = \bigcap_{i\in I} \alpha_i(x_i)$ for all $x = (x_i)_{i\in I} \in B$.*

The OR-soft set $\widetilde{\bigvee}_{i\in I}(\alpha_i, A_i)$ of the soft sets $(\alpha_i, A_i)_{i\in I}$ is defined by the soft set (γ, B) such that $B = \prod_{i\in I} A_i$, and $\gamma(x) = \bigcup_{i\in I} \alpha_i(x_i)$ for all $x = (x_i)_{i\in I} \in B$.

Note that if $A_i = A$ and $\alpha_i = \alpha$ for all $i \in I$, then $\widetilde{\bigwedge_{i\in I}}(\alpha_i, A_i)$ and $\widetilde{\bigvee_{i\in I}}(\alpha_i, A_i)$ are denoted by $\widetilde{\bigwedge_{i\in I}}(\alpha, A)$ and $\widetilde{\bigvee_{i\in I}}(\alpha, A)$ respectively. In this case, $\prod_{i\in I} A_i$ means the direct power A^I.

Theorem 3.2 *Let $(\alpha_i, A_i)_{i\in I}$ be a nonempty family of soft ternary semirings over a ternary semiring S. Then we have the following:*

(i) *$\widetilde{\bigwedge_{i\in I}}(\alpha_i, A_i)$ is a soft ternary semiring over S if it is non-null.*

(ii) *If $\{A_i \mid i \in I\}$ are pairwise disjoint, i.e., $i \neq j$ implies that $A_i \cap A_j = \phi$, then $\widetilde{\bigcup_{i\in I}}(\alpha_i, A_i)$ is a soft ternary semiring over S.*

(iii) *$\widetilde{\prod}_{i\in I}(\alpha_i, A_i)$ is a soft ternary semiring over S if it is non-null.*

Definition 3.7 *Let (η, A) be a soft ternary semiring over a ternary semiring S.*

A non-null soft set (α, I) over S is called a soft left ideal of (η, A) if

(i) *$I \subset A$,*

(ii) *$\alpha(x)$ is a left ideal of $\eta(x)$ for all $x \in \mathrm{Supp}(\alpha, I)$, i.e., $\eta(x)\eta(x)\alpha(x) \subset \alpha(x)$ for all $x \in \mathrm{Supp}(\alpha, I)$ and it is denoted by $(\alpha, I)\widetilde{\lhd_l}(\eta, A)$.*

A non-null soft set (β, I) over S is called a soft right ideal of (η, A) if

(i) $I \subset A$,

(ii) *$\beta(x)$ is a right ideal of $\eta(x)$ for all $x \in \mathrm{Supp}(\beta, I)$, i.e., $\beta(x)\eta(x)\eta(x) \subset \beta(x)$ for all $x \in \mathrm{Supp}(\beta, I)$ and it is denoted by $(\beta, I)\widetilde{\lhd_r}(\eta, A)$.*

A non-null soft set (γ, I) over S is called a soft lateral ideal of (η, A) if

(i) $I \subset A$,

(ii) *$\gamma(x)$ is a lateral ideal of $\eta(x)$ for all $x \in \mathrm{Supp}(\gamma, I)$, i.e., $\eta(x)\gamma(x)\eta(x) \subset \gamma(x)$ for all $x \in \mathrm{Supp}(\gamma, I)$ and it is denoted by $(\gamma, I)\widetilde{\lhd_m}(\eta, A)$.*

A non-null soft set (ρ, I) is called a soft ideal of (η, A) if it is left, right and lateral ideal of (η, A) and it is denoted by $(\rho, I)\widetilde{\lhd}(\eta, A)$.

Example 3.4 We consider the soft ternary semiring $(\eta, \mathbb{N}_0)$ over the ternary semiring $M_2(\mathbb{Z}_0^-)$, where $\eta : \mathbb{N}_0 \to \mathcal{P}(M_2(\mathbb{Z}_0^-))$ is given by

$$\eta(n) = \begin{pmatrix} nk_1 & nk_2 \\ nk_3 & nk_4 \end{pmatrix}$$

with $k_1, k_2, k_3, k_4 \in \mathbb{Z}_0^-$. Now we consider $(\gamma, 2\mathbb{N}_0)$, where $\gamma : 2\mathbb{N}_0 \to \mathcal{P}(M_2(\mathbb{Z}_0^-))$ is defined by

$$\gamma(m) = \begin{pmatrix} mk_1 & 0 \\ mk_2 & 0 \end{pmatrix}$$

with $m \in 2\mathbb{N}_0$ and for some $k_1, k_2 \in \mathbb{Z}_0^-$. Here $\eta(x)\eta(x)\gamma(x) \subseteq \gamma(x)$ for all $x \in \mathrm{Supp}(\gamma, 2\mathbb{N}_0)$. So $(\gamma, 2\mathbb{N}_0)$ is a soft left ideal of $(\eta, \mathbb{N}_0)$.

Example 3.5 Consider the soft ternary semiring $(\eta, \mathbb{N}_0)$ as in Example 3.4 over the ternary semiring $M_2(\mathbb{Z}_0^-)$. Now consider the soft set $(\gamma, 2\mathbb{N}_0)$ over the ternary semiring $M_2(\mathbb{Z}_0^-)$, where $\gamma : 2\mathbb{N}_0 \to \mathcal{P}(M_2(\mathbb{Z}_0^-)$ is defined by

$$\gamma(m) = \begin{pmatrix} mk_1 & mk_2 \\ 0 & 0 \end{pmatrix}$$

with $k_1, k_2 \in \mathbb{Z}_0^-$ and $m \in 2\mathbb{N}_0$. Then $\gamma(x)\eta(x)\eta(x) \subseteq \gamma(x)$ for all $x \in \mathrm{Supp}(\gamma, 2\mathbb{N}_0)$. So $(\gamma, 2\mathbb{N}_0)$ is a soft right ideal of $(\eta, \mathbb{N}_0)$.

Example 3.6 Consider the soft ternary semiring $(\eta, \mathbb{N}_0)$, the one defined in Example 3.4 over the ternary semiring $M_2(\mathbb{Z}_0^-)$. Now we consider the soft set $(\gamma, 4\mathbb{N}_0)$ over the ternary semiring $M_2(\mathbb{Z}_0^-)$. Define $\gamma : 4\mathbb{N}_0 \to \mathcal{P}(M_2(\mathbb{Z}_0^-)$ by

$$\gamma(m) = \begin{pmatrix} mk_1 & mk_2 \\ mk_3 & mk_4 \end{pmatrix},$$

where $k_1, k_2, k_3, k_4 \in \mathbb{Z}_0^-$ and $m \in 4\mathbb{N}_0$. Then $\eta(x)\gamma(x)\eta(x) \subseteq \gamma(x)$ for all $x \in \mathrm{Supp}(\gamma, 4\mathbb{N}_0)$. So $(\gamma, 4\mathbb{N}_0)$ is a soft lateral ideal of $(\eta, \mathbb{N}_0)$.

Theorem 3.3 *Let (γ_1, I_1) and (γ_2, I_2) be two soft ideals of a soft ternary semiring (η, A) over a ternary semiring S. Then the soft set $(\gamma_1, I_1)\widetilde{\sqcap}(\gamma_2, I_2)$ is a soft ideal of (η, A) if it is non-null.*

Theorem 3.4 *Let (γ, I) and (ϑ, J) be soft ideals of a soft ternary semiring (η, A) over a ternary semiring S. If I and J are disjoint, then $(\gamma, I)\widetilde{\cup}(\vartheta, J)$ is a soft ideal of (η, A).*

Theorem 3.5 *Let (η, A) be a soft ternary semiring over a ternary semiring S and $(\alpha_i, A_i)_{i \in I}$ be a nonempty family of soft ideals of (η, A). Then we have the following :*

(i) *If $\widetilde{\bigwedge_{i \in I}}(\alpha_i, A_i)$ is non-null, then $\widetilde{\bigwedge_{i \in I}}(\alpha_i, A_i)$ is a soft ideal of (η, A).*

(ii) *If $\{A_i \mid i \in I\}$ are pairwise disjoint, i.e., $i \neq j$ implies $A_i \cap A_j = \phi$, then $\widetilde{\bigcup_{i \in I}}(\alpha_i, A_i)$ is a soft ideal of (η, A).*

(iii) *$\widetilde{\bigsqcap_{i \in I}}(\alpha_i, A_i)$ is a soft ideal of (η, A) if it is non-null.*

Definition 3.8 *Let (η, A) be a non-null soft set over a ternary semiring S. Then (η, A) is called an idealistic soft ternary semiring over S if $\eta(x)$ is an ideal of S for all $x \in \mathrm{Supp}(\eta, A)$.*

Example 3.7 Consider the non-null soft set $(\eta, \mathbb{N}_0)$ over the ternary semiring $\mathbb{Z}_o^-$, where $\eta : \mathbb{N}_0 \to \mathcal{P}(\mathbb{Z}_o^-)$ is defined by $\eta(n) = n\mathbb{Z}_o^-$ for all $n \in \mathbb{N}_0$. Since $\eta(n) = n\mathbb{Z}_o^-$ is a ternary subsemiring of $\mathbb{Z}_o^-$, it is also an ideal of $\mathbb{Z}_o^-$ for all $n \in \mathbb{N}_0$. So $(\eta, \mathbb{N}_0)$ is an idealistic soft ternary semiring of the ternary semiring $\mathbb{Z}_0^-$.

Proposition 3.2 *Let (η, A) be a soft set over a ternary semiring S and $B \subseteq A$. If (η, A) is an idealistic soft ternary semiring over S, then so is (η, B), whenever it is non-null.*

Theorem 3.6 *Let (α, A) and (β, B) be two idealistic soft ternary semirings over a ternary semiring S. Then $(\alpha, A)\widetilde{\sqcap}(\beta, B)$ is an idealistic soft ternary semiring over S if it is non-null.*

Theorem 3.7 *Let (η, A) and (γ, B) be two idealistic soft ternary semirings over a ternary semiring S. If A and B are disjoint, then the union $(\eta, A)\widetilde{\cup}(\gamma, B)$ is an idealistic soft ternary soft ternary semiring over S.*

Theorem 3.8 *Let (η, A) and (γ, B) be two idealistic soft ternary semirings over a ternary semiring S. Then $(\eta, A)\widetilde{\wedge}(\gamma, B)$ is an idealistic soft ternary semiring over S if it is non-null.*

4. Soft Quasi-ideals and Soft Bi-ideals of Soft Ternary Semirings

In a ternary semiring, S. Kar [9] first introduced the notion of quasi-ideal and bi-ideal which are as follows:

Definition 4.1 [9] *An additive subsemigroup Q of a ternary semiring S is called a quasi-ideal of S if $QSS \cap (SQS + SSQSS) \cap SSQ \subseteq Q$.*

A ternary subsemiring B of a ternary semiring S is called a bi-ideal of S if $BSBSB \subseteq B$.

Now we provide these two notions in the soft set setting as follows:

Definition 4.2 *A non-null soft set (α, Q) is said to be a soft quasi-ideal of a soft ternary semiring (η, A) over a ternary semiring S if*
$((\alpha, Q)\widehat{\odot}(\eta, A)\widehat{\odot}(\eta, A))\widetilde{\sqcap}(((\eta, A)\widehat{\odot}(\alpha, Q)\widehat{\odot}(\eta, A)) \oplus ((\eta, A)\widehat{\odot}(\eta, A)\widehat{\odot}(\alpha, Q)\widehat{\odot}(\eta, A)\widehat{\odot}(\eta, A)))\widetilde{\sqcap}((\eta, A)\widehat{\odot}(\eta, A)\widehat{\odot}(\alpha, Q))\widetilde{\sqsubseteq}(\alpha, Q)$.

Theorem 4.1 *A soft set (α, Q) is a soft quasi-ideal of a soft ternary semiring (η, A) over a ternary semiring S if and only if $\alpha(x)$ is a quasi-ideal of $\eta(x)$ for all $x \in$* Supp(α, Q).

Proof First suppose that $(\alpha, Q)\widehat{\odot}(\eta, A)\widehat{\odot}(\eta, A) = (\gamma_1, I_1)$, $(\eta, A)\widehat{\odot}(\alpha, Q)\widehat{\odot}(\eta, A) = (\gamma_2, I_2)$ and $(\eta, A)\widehat{\odot}(\eta, A)\widehat{\odot}(\alpha, Q) = (\gamma_3, I_3)$. Then $\gamma_1(x) = \alpha(x)\eta(x)\eta(x)$ for all $x \in$ Supp(γ_1, I_1), where $I_1 = Q \cap A \cap A$, $\gamma_2(x) = (\eta(x)\alpha(x)\eta(x) + \eta(x)\eta(x)\alpha(x)\eta(x)\eta(x))$ for all $x \in$Supp(γ_2, I_2), where $I_2 = A \cap Q \cap A$ and $\gamma_3(x) = \eta(x)\eta(x)\alpha(x)$ for all $x \in$ Supp (γ_3, I_3), where $I_3 = A \cap A \cap Q$.

Now we suppose that $\alpha(x)$ is a quasi-ideal of $\eta(x)$ for all $x \in Supp(\alpha, Q)$. We have to show that $(\gamma_1, I_1)\widetilde{\sqcap}(\gamma_2, I_2)\widetilde{\sqcap}(\gamma_3, I_3)\widetilde{\sqsubseteq}(\alpha, Q)$. Suppose $(\gamma_1, I_1)\widetilde{\sqcap}(\gamma_2, I_2)\widetilde{\sqcap}(\gamma_3, I_3) = (\gamma_4, I_4)$. Then $\gamma_4(x) = \gamma_1(x) \cap \gamma_2(x) \cap \gamma_3(x)$ for all $x \in$ Supp(γ_4, I_4), where $I_4 = I_1 \cap I_2 \cap I_3$. Let $x \in$ Supp(γ_4, I_4). Then $x \in$ Supp(α, Q). By our assumption, $\alpha(x)$ is a quasi-ideal of $\eta(x)$. This implies that $\alpha(x)\eta(x)\eta(x) \cap (\eta(x)\alpha(x)\eta(x) + \eta(x)\eta(x)\alpha(x)\eta(x)\eta(x)) \cap \eta(x)\eta(x)\alpha(x) \subseteq \alpha(x)$ and $\gamma_1(x) \cap \gamma_2(x) \cap \gamma_3(x) \subseteq \alpha(x)$. Therefore, $\gamma_4(x) \subseteq \alpha(x)$. Also $I_4 \subseteq Q$. Consequently, $(\gamma_4, I_4)\widetilde{\sqsubseteq}(\alpha, Q)$. Hence $(\gamma_1, I_1)\widetilde{\sqcap}(\gamma_2, I_2)\widetilde{\sqcap}(\gamma_3, I_3)\widetilde{\sqsubseteq}(\alpha, Q)$ i.e., (α, Q) is a soft quasi-ideal of (η, A).

Conversely, suppose that (α, Q) is a soft quasi-ideal of a soft ternary semiring (η, A). Then $(\gamma_1, I_1)\widetilde{\sqcap}(\gamma_2, I_2)\widetilde{\sqcap}(\gamma_3, I_3)\widetilde{\sqsubseteq}(\alpha, Q)$ i.e., $(\gamma_4, I_4)\widetilde{\sqsubseteq}(\alpha, Q)$, where $I_4 = I_1 \cap I_2 \cap I_3$. This implies that $\gamma_4(x) \subseteq \alpha(x)$ for all $x \in$ Supp (γ_4, I_4). Now $\gamma_4(x) \subseteq \alpha(x) \Longrightarrow \gamma_1(x) \cap \gamma_2(x) \cap \gamma_3(x) \subseteq \alpha(x) \Longrightarrow \alpha(x)\eta(x)\eta(x) \cap (\eta(x)\alpha(x)\eta(x) + \eta(x)\eta(x)\alpha(x)\eta(x)\eta(x)) \cap \eta(x)\eta(x)\alpha(x) \subseteq \alpha(x)$. This shows that $\alpha(x)$ is a quasi-ideal of $\eta(x)$ since $x \in$ Supp$(\gamma_4, I_4) \Longrightarrow x \in$ Supp(α, Q). Hence $\alpha(x)$ is a quasi-ideal of $\eta(x)$ for all $x \in$ Supp(α, Q).

Example 4.1 Consider the soft ternary semiring $(\eta, \mathbb{N}_0)$, the one defined in Example 3.4 over the ternary semiring $M_2(\mathbb{Z}_0^-)$. Now we consider the soft set $(\gamma, 2\mathbb{N}_0)$ over the ternary semiring $(\eta, \mathbb{N}_0)$, where $\gamma : 2\mathbb{N}_0 \to \mathcal{P}(M_2(\mathbb{Z}_0^-))$ is defined by

$$\gamma(m) = \begin{pmatrix} mk_1 & 0 \\ 0 & 0 \end{pmatrix}$$

with $k_1 \in \mathbb{Z}_0^-$ and $m \in$ Supp $(\gamma, 2\mathbb{N}_0)$. Here it is easy to verify that $\gamma(x)$ is a quasi-ideal of $\eta(x)$ for all $x \in$ Supp$(\gamma, 2\mathbb{N}_0)$. So $(\gamma, 2\mathbb{N}_0)$ is a soft quasi-ideal of the soft ternary semiring $(\eta, \mathbb{N}_0)$ over the ternary semiring $M_2(\mathbb{Z}_0^-)$. But it is not a soft left ideal, a soft lateral ideal or a soft right ideal of $(\eta, \mathbb{N}_0)$.

Definition 4.3 *A soft ternary subsemiring (α, B) of a soft ternary semiring (η, A) over a ternary semiring S is said to be a soft bi-ideal of (η, A) if*

$(\alpha, B)\widehat{\odot}(\eta, A)\widehat{\odot}(\alpha, B)\widehat{\odot}(\eta, A)\widehat{\odot}(\alpha, B) \sqsubseteq (\alpha, B)$.

Theorem 4.2 *A soft set (α, B) is a soft bi-ideal of a soft ternary semiring (η, A) over ternary semiring S if and only if $\alpha(x)$ is a bi-ideal of $\eta(x)$ for all $x \in \mathrm{Supp}(\alpha, B)$.*

Proof The proof is similar to the proof of Theorem 4.1.

Example 4.2 Consider the soft set $(\eta, \mathbb{N}_0)$ over the ternary semiring

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{pmatrix} : a, b, c, d, e \in \mathbb{Z}_0^- \right\},$$

where $\eta : \mathbb{N}_0 \to \mathcal{P}(S)$ is defined by

$$\eta(n) = \begin{pmatrix} na & nb & nc \\ 0 & 0 & nd \\ 0 & 0 & ne \end{pmatrix}$$

for all $n \in \mathbb{N}_0$ and $a, b, c, d, e \in \mathbb{Z}_0^-$. Now we consider the soft set $(\gamma, 2\mathbb{N}_0)$ over S, where $\gamma : 2\mathbb{N}_0 \to \mathcal{P}(S)$ is defined by

$$\gamma(m) = \begin{pmatrix} 0 & mk & 0 \\ 0 & 0 & mk \\ 0 & 0 & 0 \end{pmatrix}$$

for all $m \in 2\mathbb{N}_0$ and $k \in \mathbb{Z}_0^-$. Since $2\mathbb{N}_0 \subseteq \mathbb{N}_0$ and $\gamma(x)\eta(x)\gamma(x)\eta(x)\gamma(x) \subseteq \gamma(x)$ for all $x \in \mathrm{Supp}(\gamma, 2\mathbb{N}_0)$. So $(\gamma, 2\mathbb{N}_0)$ is a soft bi-ideal of $(\eta, \mathbb{N}_0)$ by Theorem 4.2.

Theorem 4.3 *Every soft quasi ideal (α, Q) of a soft ternary semiring (η, A) over a ternary semiring S is a soft bi-ideal of (η, A).*

Proof Let (α, Q) be a soft quasi-ideal of a soft ternary semiring (η, A) over S. Then $\alpha(x)$ is a quasi-ideal of $\eta(x)$, where $x \in \mathrm{Supp}(\alpha, Q)$. Thus for all $x \in \mathrm{Supp}(\alpha, Q)$, we find that $\alpha(x)\eta(x)\eta(x) \cap (\eta(x)\alpha(x)\eta(x) + \eta(x)\eta(x)\alpha(x)\eta(x)\eta(x)) \cap \eta(x)\eta(x)\alpha(x) \subseteq \alpha(x)$. Clearly, we see that $\alpha(x)\eta(x)\alpha(x)\eta(x)\alpha(x) \subseteq \alpha(x)\eta(x)\eta(x)$ and $\alpha(x)\eta(x)\alpha(x)\eta(x)\alpha(x) \subseteq \eta(x)\eta(x)\alpha(x)$. Also $\{0\} \subseteq \eta(x)\alpha(x)\eta(x)$ and $\alpha(x)\eta(x)\alpha(x)\eta(x)\alpha(x) \subseteq \eta(x)\eta(x)\alpha(x)\eta(x)$ $\eta(x)$. So we find that $\alpha(x)\eta(x)\alpha(x)\eta(x)\alpha(x) \subseteq (\eta(x)\alpha(x)\eta(x) + \eta(x)\eta(x)\alpha(x)\eta(x)\eta(x))$. Thus it follows that $\alpha(x)\eta(x)\alpha(x)\eta(x)\alpha(x) \subseteq \alpha(x)\eta(x)\eta(x) \cap (\eta(x)\alpha(x)\eta(x) + \eta(x)\eta(x)$ $\alpha(x)\eta(x)\eta(x)) \cap \eta(x)\eta(x)\alpha(x) \subseteq \alpha(x)$. This shows that $\alpha(x)$ is a bi-ideal of $\eta(x)$ for all $x \in \mathrm{Supp}(\alpha, Q)$ and hence (α, Q) is a soft bi-ideal of (η, A).

Note 4.1 The converse of the above result is not true, i.e., a soft bi-ideal (β, B) of a soft ternary semiring (η, A) over a ternary semiring S is not a soft quasi-ideal of (η, A), in general. But we shall see later that these two notions coincide in a certain class of ternary semirings.

Example 4.3 In Example 4.2, we defined the bi-ideal $(\gamma, 2\mathbb{N}_0)$ which is not a quasi-ideal. Because if we consider $P, Q, R, S \in \eta(1)$, where

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Q = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \; S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and $T = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \in \gamma(1)$, then $PQT = U = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $PTR = U$ also $TRS = U$.

So $U \in \Big(\eta(1)\eta(1)\gamma(1) \cap \big((\eta(1)\gamma(1)\eta(1)) + (\eta(1)\eta(1)\gamma(1)\eta(1)\eta(1))\big) \cap \gamma(1)\eta(1)\eta(1)\Big)$. But $U \notin \gamma(1)$. So $\gamma(1)$ is not a quasi-ideal of $\eta(1)$. Hence $(\gamma, 2\mathbb{N}_0)$ is not a soft quasi-ideal of $(\eta, \mathbb{N}_0)$.

Theorem 4.4 *Any left, right or lateral soft ideal of a soft ternary semiring over a ternary semiring is a soft quasi-ideal.*

Proof Let us assume that (α, L), (β, R) and (γ, M) are the soft left, soft right and soft lateral ideal of a soft ternary semiring (η, A) over a ternary semiring S respectively.

Now for all $x \in \mathrm{Supp}(\alpha, L)$, we see that

$\alpha(x)\eta(x)\eta(x) \cap (\eta(x)\alpha(x)\eta(x) + \eta(x)\eta(x)\alpha(x)\eta(x)\eta(x)) \cap \eta(x)\eta(x)\alpha(x) \subseteq \eta(x)\eta(x)\alpha(x)$ $\subseteq \alpha(x)$ since (α, L) is a soft left ideal of (η, A).

Similarly, for all $x \in \mathrm{Supp}(\beta, R)$, we see that

$\beta(x)\eta(x)\eta(x) \cap (\eta(x)\beta(x)\eta(x) + \eta(x)\eta(x)\beta(x)\eta(x)\eta(x)) \cap \eta(x)\eta(x)\beta(x) \subseteq \beta(x)\eta(x)\eta(x)$ $\subseteq \beta(x)$ since (β, R) is a soft right ideal of (η, A).

Also for all $x \in \mathrm{Supp}(\gamma, M)$, we find that

$\gamma(x)\eta(x)\eta(x) \cap (\eta(x)\gamma(x)\eta(x) + \eta(x)\eta(x)\gamma(x)\eta(x)\eta(x)) \cap \eta(x)\eta(x)\gamma(x) \subseteq \eta(x)\gamma(x)\eta(x)$ $\subseteq \gamma(x)$ since (γ, M) is a soft lateral ideal of (η, A).

Thus from above, it follows that (α, L), (β, R) and (γ, M) are soft quasi-ideals of (η, A).

Theorem 4.5 *Let (α, L), (β, M) and (γ, R) be a soft left, a soft lateral and a soft right ideal of a soft ternary semiring (η, A) respectively over a ternary semiring S. Then their bi-intersection is also a soft quasi-ideal of (η, A).*

Proof Let $(\alpha, L)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\gamma, R) = (\delta, K)$, where $K = L \cap M \cap R$ and $\delta(x) = \alpha(x) \cap \beta(x) \cap \gamma(x)$ for all $x \in \mathrm{Supp}(\delta, A)$. Now $\delta(x)\eta(x)\eta(x) = (\alpha(x) \cap \beta(x) \cap \gamma(x))\eta(x)\eta(x) \subseteq \gamma(x)\eta(x)\eta(x) \subseteq \gamma(x)$ since (γ, R) is a soft right ideal of (η, A). Similarly, $\eta(x)\eta(x)\delta(x) \subseteq \alpha(x)$ since (α, L) is a soft left ideal of (η, A) and $\eta(x)\delta(x)\eta(x) + \eta(x)\eta(x)\delta(x)\eta(x)\eta(x) \subseteq \eta(x)\beta(x)\eta(x) \subseteq \beta(x)$. Thus it follows that $\delta(x)\eta(x)\eta(x) \cap (\eta(x)\delta(x)\eta(x) + \eta(x)\eta(x)\delta(x)\eta(x)\eta(x)) \cap \eta(x)\eta(x)\delta(x) \subseteq \alpha(x) \cap \beta(x) \cap \gamma(x) = \delta(x)$. Hence (δ, K) is a soft quasi-ideal of (η, A).

5. Regularity in Soft Ternary Semirings

Definition 5.1 *A soft ternary semiring (η, A) over a ternary semiring S is said to be regular if $\eta(x)$ is regular for all $x \in \mathrm{Supp}(\eta, A)$.*

Example 5.1 We consider the ternary semiring $\mathbb{Z}_0^-$ with the binary operation called addition defined by $a \oplus b = \max\{a, b\}$ and the ternary multiplication defined by $a \otimes b \otimes c = \min\{a, b, c\}$. It can be verified that $S = (\mathbb{Z}_0^-, \oplus, \otimes)$ is a regular ternary semiring. Now we consider the soft set $(\eta, \mathbb{N}_0)$ over the ternary semiring S, where $\eta : \mathbb{N}_0 \to \mathcal{P}(S)$ is defined by $\eta(n) = n\mathbb{Z}_0^-$ for all $n \in \mathbb{N}_0$. We can verify that $\eta(n)$ is regular for all $n \in \mathrm{Supp}(\eta, \mathbb{N}_0)$. So $(\eta, \mathbb{N}_0)$ is a regular soft ternary semiring over the ternary semiring S.

Theorem 5.1 *Let (η, A) be a soft ternary semiring over a ternary semiring S. Then (η, A) is a regular soft ternary semiring if and only if $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L) = (\gamma, R)$ $\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L)$ for all soft left ideal (α, L), soft lateral ideal (β, M) and soft right ideal (γ, R) of (η, A).*

Proof First suppose that (η, A) is a regular soft ternary semiring over a ternary semiring S. Let (α, L), (β, M) and (γ, R) be a soft left, a soft lateral and a soft right ideal of (η, A) respectively. Let $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L) = (\delta, K)$, where $K = R \cap M \cap L$ and $\delta(x) = \gamma(x) \cap \beta(x) \cap \alpha(x)$ for all $x \in \mathrm{Supp}(\delta, K)$. Now suppose $(\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L) = (\theta, J)$, where $J = R \cap M \cap L$ and $\theta(x) = \gamma(x)\beta(x)\alpha(x)$ for all $x \in$ Supp (θ, J). If $x \in \mathrm{Supp}(\delta, K)$, then $\delta(x) = \gamma(x) \cap \beta(x) \cap \alpha(x) = \gamma(x)\beta(x)\alpha(x) = \theta(x)$ since (η, A) is a regular soft ternary semiring. This shows that $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L) \subseteq (\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L)$. Again if $x \in$Supp(θ, J), then $\theta(x) = \gamma(x)\beta(x)\alpha(x) = \gamma(x) \cap \beta(x) \cap \alpha(x) = \delta(x)$ since (η, A) is a regular soft ternary semiring. This implies that $(\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L) \subseteq (\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L)$. Consequently, we find that $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L) = (\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L)$.

Conversely, suppose that $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L) = (\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L)$. Then $L = R = M = \eta(x)$, where $x \in A$. Now we define $\beta : \eta(x) \mapsto \mathcal{P}(\eta(x))$ by $\beta(u) = u\eta(u)\eta(u) + nu$, where $n \in \mathbb{Z}_0^+$ and $u \in \eta(x)$, define $\gamma : \eta(x) \mapsto \mathcal{P}(\eta(x))$ by $\gamma(u) = \eta(u)u\eta(u) + \eta(u)\eta(u)u\eta(u)\eta(u) + nu$, where $n \in \mathbb{Z}_0^+$ and $u \in \eta(x)$ and we define $\alpha : \eta(x) \mapsto \mathcal{P}(\eta(x))$ by $\alpha(u) = \eta(u)\eta(u)u + nu$, where $n \in \mathbb{Z}_0^+$ and $u \in \eta(x)$. Then $(\beta, \eta(x))$ is a soft right ideal of (η, A), $(\gamma, \eta(x))$ is a lateral ideal of (η, A) and $(\alpha, \eta(x))$ is a soft left ideal of (η, A) over $\eta(x)$. Thus $u \in \beta(u) \cap \gamma(u) \cap \alpha(u) = \beta(u)\gamma(u)\alpha(u) = (u\eta(u)\eta(u) + nu)(\eta(u)u\eta(u) + \eta(u)\eta(u)u\eta(u)\eta(u) + nu)(\eta(u)\eta(u)u + nu) \subseteq (u\eta(u)\eta(u) + nu)(\eta(u)u\eta(u) + nu)(\eta(u)\eta(u)u + nu) \subseteq u\eta(u)u$. So $u \in u\eta(u)u$. Hence $\eta(x)$ is regular for all $x \in \mathrm{Supp}(\eta, A)$.

Definition 5.2 *A soft subset (α, B) of a soft ternary semiring (η, A) over a ternary semiring S is called idempotent if $(\alpha, A)\widehat{\odot}(\alpha, A)\widehat{\odot}(\alpha, A) = (\alpha, A)$.*

Theorem 5.2 *If for every soft quasi-ideal (θ, Q) of a soft ternary semiring (η, A) over a ternary semiring S is idempotent, then (η, A) is a regular soft ternary semiring.*

Proof Let (α, L), (β, M) and (γ, R) be a soft left, a soft lateral and a soft right ideal of (η, A) respectively. Then their bi-intersection is a soft quasi-ideal of (η, A). Now by the assumption, $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L) = ((\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L))\widehat{\odot}((\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L))$ $\widehat{\odot}$ $((\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L)) \subseteq (\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L)$. Let $(\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L) = (\delta, I)$. Then $\delta(x) = \gamma(x)\beta(x)\alpha(x)$ for all $x \in \mathrm{Supp}(\delta, I)$, where $I = R \cap M \cap L$. Also $\gamma(x)$ is a right ideal of $\eta(x)$ for all $x \in \mathrm{Supp}(\gamma, R)$, $\beta(x)$ is a soft lateral ideal of $\eta(x)$ for all $x \in \mathrm{Supp}(\beta, M)$ and $\alpha(x)$ is a left ideal of $\eta(x)$ for all $x \in \mathrm{Supp}(\alpha, L)$. Thus for all

$x \in$Supp(δ, I), we have $\gamma(x)\beta(x)\alpha(x) \subseteq \gamma(x)\cap\beta(x)\cap\alpha(x)$. So $(\delta, I)\widetilde{\sqsubseteq}(\gamma, R)\widetilde{\sqcap}(\gamma, M)\widetilde{\sqcap}(\alpha, L)$. Thus $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L) = (\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L)$. Hence (η, A) is a regular soft ternary semiring by Theorem 5.1.

Theorem 5.3 *A soft ternary semiring (η, A) over a commutative ternary semiring S is regular if and only if every soft ideal (α, I) of (η, A) is idempotent.*

Proof Suppose that (η, A) is regular and (α, I) is a soft ideal of (η, A). Now $(\alpha, I) = (\alpha, I)\widetilde{\sqcap}(\alpha, I)\widetilde{\sqcap}(\alpha, I) = (\alpha, I)\widehat{\odot}(\alpha, I)\widehat{\odot}(\alpha, I)$ since (η, A) is regular. So (α, I) is idempotent.

Conversely, suppose that every soft ideal of (η, A) is idempotent. Let (α, L), (β, M) and (γ, R) be a soft left ideal, a soft lateral ideal and a soft right ideal of (η, A) respectively. Since S is commutative so (α, L), (β, M) and (γ, R) are all soft ideals of (η, A). Thus it follows that $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L)$ is a soft ideal of (η, A). So by our assumption, we have $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L) = ((\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L))\widehat{\odot}((\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L))\widehat{\odot}$ $((\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L))\widetilde{\sqsubseteq}(\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L)$. Thus $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L)\widetilde{\sqsubseteq}(\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L)$. The reverse inclusion is obvious. Consequently, $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L) = (\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L)$ and hence (η, A) is regular by Theorem 5.2.

Theorem 5.4 *Let (η, A) be a regular soft ternary semiring over a ternary semiring S. Then every soft bi-ideal (α, B) of (η, A) is a soft quasi-ideal of (η, A).*

Proof Let (α, B) be a soft bi-ideal of a regular soft ternary semiring (η, A) over a ternary semiring S. Then $\alpha(x)$ is a bi-ideal of $\eta(x)$ for all $x \in$ Supp(α, B). Now $\alpha(x)\eta(x)\eta(x) \cap (\eta(x)\alpha(x)\eta(x) + \eta(x)\eta(x)\alpha(x)\eta(x)\eta(x)) \cap \eta(x)\eta(x)\alpha(x) = \alpha(x)\eta(x)\eta(x)$ $(\eta(x)\alpha(x)\eta(x)+\eta(x)\eta(x)\alpha(x)\eta(x)\eta(x))\,\eta(x)\eta(x)\alpha(x)$ (since (η, A) is regular) $\subseteq \alpha(x)\eta(x)$ $\alpha(x)\eta(x)\alpha(x) \subseteq \alpha(x)$ (since $\alpha(x)$ is a bi-ideal of $\eta(x)$). Thus $\alpha(x)$ is a quasi-ideal of $\eta(x)$. Hence (α, B) is a soft quasi-ideal of (η, A).

Definition 5.3 *A soft ternary semiring (η, A) over a ternary semiring S is said to be intra-regular if $\eta(x)$ is intra-regular for all $x \in$ Supp(η, A).*

Theorem 5.5 *A soft ternary semiring (η, A) over a ternary semiring S is intra-regular if and only if $(\alpha, L)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\gamma, R) \sqsubseteq (\alpha, L)\widehat{\odot}(\beta, M)\widehat{\odot}(\gamma, R)$.*

Proof Suppose (η, A) is intra-regular. Let (α, L), (β, M) and (γ, R) be a soft left ideal, soft lateral ideal and a soft right ideal of (η, A) respectively. Let $(\alpha, L)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}$ $(\gamma, R) = (\delta_1, I_1)$ and $(\alpha, L)\widehat{\odot}(\beta, M)\widehat{\odot}(\gamma, R) = (\delta_2, I_2)$. Then $\delta_1(x) = \alpha(x) \cap \beta(x) \cap \gamma(x)$ for all $x \in$ Supp (δ_1, I_1), where $I_1 = L \cap M \cap R$ and $\delta_2 = \alpha(x)\beta(x)\gamma(x)$ for all $x \in$ Supp(δ_2, I_2), where $I_2 = L \cap M \cap R$. Let $x \in L \cap M \cap R$. Then $\alpha(x)$ is a left ideal of $\eta(x)$ since (α, L) is a soft left ideal of (η, A). Similarly, $\beta(x)$ is a lateral ideal of $\eta(x)$ and $\gamma(x)$ is a right ideal of $\eta(x)$. Now since $\eta(x)$ is intra-regular, we have $\alpha(x) \cap \beta(x) \cap \gamma(x) \subseteq \alpha(x)\beta(x)\gamma(x)$. Thus it follows that $\delta_1(x) \subseteq \delta_2(x)$ and hence $(\delta_1, I_1) \sqsubseteq (\delta_2, I_2)$. Thus $(\alpha, L)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\gamma, R) \sqsubseteq (\alpha, L)\widehat{\odot}(\beta, M)\widehat{\odot}(\gamma, R)$.

Conversely, let $(\alpha, L)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\gamma, R) \sqsubseteq (\alpha, L)\widehat{\odot}(\beta, M)\widehat{\odot}(\gamma, R)$ for any soft left ideal (α, L), soft lateral ideal (β, M) and soft right ideal (γ, R) of (η, A). Let $L = R = M = \eta(x)$, where $x \in A$. Now we define $\beta : \eta(x) \mapsto \mathcal{P}(\eta(x))$ by $\beta(u) = u\eta(u)\eta(u) + nu$, where $n \in \mathbb{Z}_0^+$ and $u \in \eta(x)$, define $\gamma : \eta(x) \mapsto \mathcal{P}(\eta(x))$ by $\gamma(u) = \eta(u)u\eta(u) + \eta(u)\eta(u)u\eta(u)\eta(u) + nu$, where $n \in \mathbb{Z}_0^+$ and $u \in \eta(x)$ and we define $\alpha : \eta(x) \mapsto \mathcal{P}(\eta(x))$

by $\alpha(u) = \eta(u)\eta(u)u + nu$, where $n \in \mathbb{Z}_0^+$ and $u \in \eta(x)$. Then $(\beta, \eta(x))$ is a soft right ideal of (η, A) and $(\gamma, \eta(x))$ is a lateral ideal of (η, A) $(\alpha, \eta(x))$ is a soft left ideal of (η, A) over $\eta(x)$. Now $u \in \alpha(u) \cap \gamma(u) \cap \beta(u) \subseteq \alpha(u)\gamma(u)\beta(u) = (u\eta(u)\eta(u) + nu)(\eta(u)u\eta(u)+\eta(u)\eta(u)u\eta(u)\eta(u)+nu)(\eta(u)\eta(u)u+nu) \subseteq (u\eta(u)\eta(u)+nu)(\eta(u)u\eta(u)+nu)(\eta(u)\eta(u)u + nu) \subseteq \eta(u)u^3\eta(u)$. Thus $u \in \eta(u)u^3\eta(u)$. So $\eta(x)$ is intra-regular for all $x \in \mathrm{Supp}(\eta, A)$.

Theorem 5.6 *A soft ternary semiring (η, A) over a ternary semiring S is both soft regular and soft intra-regular if and only if every soft quasi-ideal (θ, Q) of (η, A) is idempotent.*

Proof Let (η, A) be regular and intra-regular over the ternary semiring S. We consider (γ, R), (β, M) and (α, L) are soft right, lateral and left ideal of (η, A) respectively. Now $(\theta, Q) = (\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L)$ is a soft quasi-ideal of (η, A). Then we find that $(\theta, Q)\widehat{\odot}(\theta, Q)\widehat{\odot}(\theta, Q) = (\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L)\widehat{\odot}(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L)\widehat{\odot}(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L)\widetilde{\sqsubseteq}(\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L) = (\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L) = (\theta, Q)$ since (η, A) is a soft regular ternary semiring. Thus it follows that $(\theta, Q)\widetilde{\odot}(\theta, Q)\widehat{\odot}(\theta, Q)\widetilde{\sqsubseteq}(\theta, Q)$. Clearly, $(\theta, Q)\widetilde{\sqsubseteq}(\theta, Q)\widehat{\odot}(\theta, Q)\widetilde{\odot}(\theta, Q)$. So (θ, Q) is idempotent.

Conversely, suppose that every quasi-ideal (θ, Q) of (η, A) is idempotent. Let (γ, R), (β, M), (α, L) be a soft right, soft lateral, soft left ideal of (η, A) respectively. Then their bi-intersection is also a soft quasi-ideal of (η, A). Thus by the assumption, it follows that $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L) = ((\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L))\widehat{\odot}((\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L))\widehat{\odot}((\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L)) \subseteq (\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L)$. Now suppose $(\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L) = (\delta, I)$. Then $\delta(x) = \gamma(x)\beta(x)\alpha(x)$ for all $x \in \mathrm{Supp}\ (\delta, I)$, where $I = R \cap M \cap L$ and $\gamma(x)$ is a right ideal of $\eta(x)$ for all $x \in \mathrm{Supp}(\gamma, R)$, $\beta(x)$ is a lateral ideal of $\eta(x)$ for all $x \in \mathrm{Supp}(\beta, M)$ and $\alpha(x)$ is a left ideal of $\eta(x)$ for all $x \in \mathrm{Supp}\ (\alpha, L)$. So for all $x \in \mathrm{Supp}(\delta, I)$, $\gamma(x)\beta(x)\alpha(x) \subseteq \gamma(x) \cap \beta(x) \cap \alpha(x)$. This implies that $\delta(x) \subseteq \gamma(x)\beta(x)\alpha(x)$. So $(\delta, I)\widetilde{\subseteq}(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L)$. Thus $(\gamma, R)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\alpha, L) = (\gamma, R)\widehat{\odot}(\beta, M)\widehat{\odot}(\alpha, L)$. Hence (η, A) is a soft regular ternary semiring. Now to show that (η, A) is intra-regular, suppose (α, L), (β, M) and (γ, R) are a soft left, a soft lateral and a soft right ideal of (η, A) respectively. Then $(\theta, Q) = (\alpha, L)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\gamma, R)$ is a quasi-ideal of (η, A) and hence it is idempotent by our assumption. Therefore $(\theta, Q)\widehat{\odot}(\theta, Q)\widehat{\odot}\ (\theta, Q) = (\theta, Q)$. Thus it follows that $(\theta, Q) = (\alpha, L)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\gamma, R) = (\alpha, L)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\gamma, R)\widehat{\odot}\ (\alpha, L)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\gamma, R)\widehat{\odot}(\alpha, L)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}\ (\gamma, R)\widetilde{\sqsubseteq}(\alpha, L)\widehat{\odot}(\beta, M)\widehat{\odot}(\gamma, R)$. So we have $(\alpha, L)\widetilde{\sqcap}(\beta, M)\widetilde{\sqcap}(\gamma, R)\widetilde{\sqsubseteq}(\alpha, L)\widehat{\odot}(\beta, M)\widehat{\odot}(\gamma, R)$. Hence (η, A) is intra-regular, by Theorem 5.5.

6. Conclusion

The results in ordinary (soft) semirings may be extended to n-ary (soft) semirings for arbitrary n but the transition from $n = 3$ to arbitrary n entails a great degree of complexity that makes it undesirable for exposition. For this reason, we shall confine ourselves in the present paper wholly to soft ternary semirings. In this paper, we introduce the notion of soft ternary semiring and some soft ideals of soft ternary semirings. Then we characterize regularity and intra-regularity in soft ternary semirings by using different soft ideals. This study can be extended to several classes of ternary algebras and may be obtained some interesting results of ternary algebra in the context of soft set theory.

Acknowledgments

The authors are extremely grateful to the anonymous referees for their constructive comments to improve the presentation of this paper.

The first author (S. Kar) is thankful to the Department of Science and Technology (DST), Government of India, for Promotion of University Research and Scientific Excellence (PURSE), 2nd Phase for providing financial assistance to carry out this research work.

References

[1] U. Acar, F. Koyuncu, B. Tanay, Soft sets and soft rings, Computers and Mathematics with Applications 59 (2010) 3458-3463.
[2] H. Aktas, N. Cagman, Soft set and soft groups, Information Sciences 177 (2007) 2726-2735.
[3] M.I. Ali, M. Shabir, K.P. Shum, On soft ideals over semigroups, Southeast Asian Bulletin Of Mathematics 34 (2010) 595-610.
[4] A.O. Atagun, A. Sezgin, Soft substructures of rings, fields and modules, Computers and Mathematics with Applications 61 (2011) 592-601.
[5] T.K. Dutta, S. Kar, On regular ternary semirings, Advances in Algebra, Proceedings of the ICM Satellite Conference in Algebra and Related Topics, World Scientific, 2003, pp. 343-355.
[6] T.K. Dutta, S. Kar, A note on regular ternary semirings, Kyungpook Mathematical Journal 46(3) (2006) 357-365.
[7] T.K. Dutta, S. Kar, K. Das, Power ternary semirings, Afrika Matematika; Springer 26 (2015) 1483-1494.
[8] F. Feng, Y.B. Jun, X. Zhao, Soft semirings, Computer and Mathematics with Applications 56 (2008) 2621-2628.
[9] S. Kar, On quasi-ideals and bi-ideals of ternary semirings, International Journal of Mathematics and Mathematical Sciences 2005(18) (2005) 3015-3023.
[10] S. Kar, Ideal theory in the ternary semiring Z_0^-, Bull. Malays. Math. Sci. Soc. (2) 34(1) (2011) 69-77.
[11] W.G. Lister, Ternary rings, Trans. Amer. Math. Soc. 154 (1971) 37-55.
[12] P.K. Maji, A.R. Roy, R. Biswas, An application of soft sets in decision making problems, Computers and Mathematics with Applications 44 (2002) 1077-1083.
[13] P.K. Maji, R. Biswas, A.R. Roy, Soft set theory, Computers and Mathematics with Applications 45 (2003) 555-562.
[14] D. Molodtsov, Soft set theory-first results, Computers and Mathematics with Applications 37 (1999) 19-31.
[15] A.S. Sezer, A new view to ring theory via soft union rings, ideals and bi-ideals, Knowledge-Based Systems 36 (2012) 300-314.
[16] A. Sezgin, A.O. Atagun, Soft groups and normalistic soft groups, Computers and Mathematics with Applications 62 (2011) 685-698.
[17] J. Wang, M. Yin, W. Gu, Soft polygroups, Computers and Mathematics with Applications 62 (2011) 3529-3537.
[18] L.A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338-353.